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FREQUENCY DEPENDENCE OF THE RVE SIZE IN RANDOM VISCOELASTIC MEDIA

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Somehow the soil is a random media..let us see in which sense

- In a given layer of soil, voids may be present;
- their distribution is random;
- one way to model the surrounding soil is top consider it viscoelastic;
- so, even if the soil is normally layered, in a given layer the matter behaves like a random viscoelastic composite;
- In finding the effective properties of the soil, already the viscoelasticity of it leads to a wide oper problem, which has been solved by the authors;
- A more realistic situation should account for the presence of water, so both saturaded and partially saturaded viscoelastic media with voids should be further examined.

Outline

What's out there:

-Nothing for random composites and their response;

-No investigations on the effects of past histories in non-virgin composites (even conventional, not just random).

What can we do? Provide insights on both aspects above. How?

1) "Smart" choice of the polarization stress w.r.to the comparison solid

2) Ensemble average (up to two-point statistics) - Hashin & Shtrickman v.p.

Integral equation for the averaged actual polarization

Nonlocal stress-strain/state response

Second-gradient approximation

First consequences on the frequency dependence of the RVE size

And alot more….

What needs to be done? A lot of work!

2 AN IMPORTANT REMARK OF DETERMINISTIC (LINEAR) VISCOELASTICITY

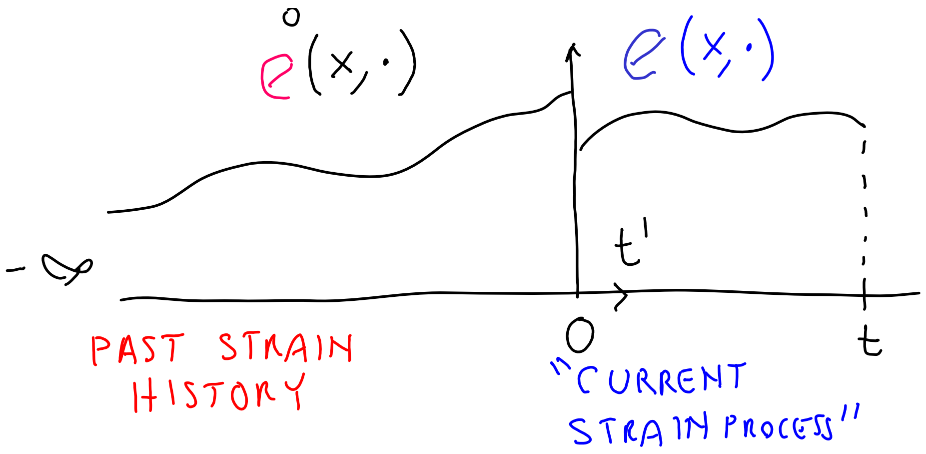
Stress $\boldsymbol{\sigma}$ at a point $\mathbf{x}$ and at a time $t \geq 0$:

$$\boldsymbol{\sigma}(\mathbf{x}, t) = \mathbb{G}(\mathbf{x}, 0)\boldsymbol{e}(\mathbf{x}, t) + \int_0^t \dot{\mathbb{G}}(\mathbf{x}, t - t')\, \boldsymbol{e}(\mathbf{x}, t')dt' + \mathbf{I}^0(\mathbf{x}, t), \tag{1}$$

where

$$\mathbf{I}^0(\mathbf{x}, t) := \int_0^{+\infty} \dot{\mathbb{G}}(\mathbf{x}, t + s)\, \boldsymbol{e}(\mathbf{x}, s)ds. \tag{2}$$

The former provides information about the state of the material at time $t \geq 0$.



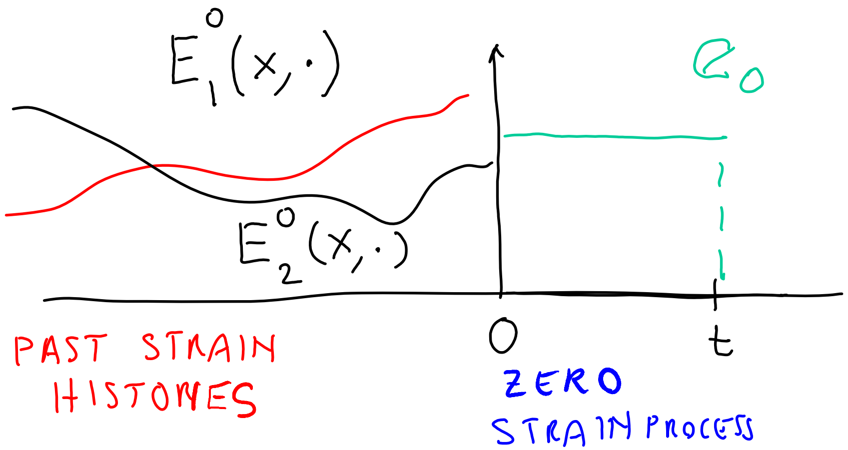
RELAXATION TESTS I

Two tests could be compared under the same prescribed constant strain $\boldsymbol{e}_0$.

The first experiment could be done by imposing such a strain on a specimen extracted by a sample of the given material.

In this first case:

$$\begin{aligned}\boldsymbol{\sigma}^0(\mathbf{x},t) &:= \mathbb{G}(\mathbf{x},0)\boldsymbol{e}_0 + \int_0^t \dot{\mathbb{G}}(\mathbf{x},t-t')\,dt'\,\boldsymbol{e}_0 + \mathbf{I}^0(\mathbf{x},t) \\ &= \mathbb{G}(\mathbf{x},t)\boldsymbol{e}(0) + \mathbf{I}^0(\mathbf{x},t), \end{aligned} \tag{3}$$



3 RELAXATION TESTS II

Annealing occurs $\Rightarrow$ *virgin state* i.e. $\mathbf{I}^0 \equiv \mathbf{0}$, hence:

$$\boldsymbol{\sigma}^{(I)}(\mathbf{x}, t) \quad := \quad \mathbb{G}(\mathbf{x}, t)\boldsymbol{e}(0) \tag{4}$$

which indeed would allow for the regular relaxation test, i.e. the determination of the relaxation function for the annealed material.

$$\mathbf{I}^0(\mathbf{x}, t) = \boldsymbol{\sigma}^0(\mathbf{x}, t) - \boldsymbol{\sigma}^{(I)}(\mathbf{x}, t)$$

$\mathbf{I}^0(\mathbf{x}, t)$ is the *State Variable*
i.e.
the *relaxing residual stress* induced by past histories experienced at x, whenever $\boldsymbol{e}(\mathbf{x}, 0) = \mathbf{0}$.

So…how can we proceed to get the effective constitutive properties of a random viscoelastic media?

Following a strategy undertaken by R. Hill (seventies), Willis (eighties and early nineties) in deterministic elastic media & Drugan and Willis (96) for the random case, we:

1) Choose a comparison solid, whose properties (either relaxation or creep Function) are spatially constant, i.e. it is homogeneous and viscoelastic;

2) Examine the deviation of the real stress from the one experienced by this fictitious body, i.e. the *polarization stress*;
Since for the given medium there is more than just one choice for it, a "smart" choice is made;

Deterministic heterogeneous viscoelastic media

1) Actual Polarization Stress

$$\boldsymbol{\tau}(\mathbf{x},t) := \boldsymbol{\Sigma}(\mathbf{x},t) - \boldsymbol{\Sigma}_0(\mathbf{x},t), \tag{10}$$

$$\boldsymbol{\Sigma}(\mathbf{x},t) := \mathbb{G}(\mathbf{x},0)\boldsymbol{e}(\mathbf{x},t) + \int_0^t \dot{\mathbb{G}}(\mathbf{x},t-t')\,\boldsymbol{e}(\mathbf{x},t')dt', \tag{11}$$

$$\boldsymbol{\Sigma}_0(\mathbf{x},t) := \mathbb{G}_0(0)\boldsymbol{e}(\mathbf{x},t) + \int_0^t \dot{\mathbb{G}}_0(t-t')\,\boldsymbol{e}(\mathbf{x},t')dt'. \tag{12}$$

2) Equivalent Expressions for the Stress

$$\begin{aligned}\boldsymbol{\sigma}(\mathbf{x},t) &= \boldsymbol{\Sigma}_0(\mathbf{x},t) + \mathbf{P}(\mathbf{x},t) + \mathbf{I}_0^0(\mathbf{x},t)\\ &= \boldsymbol{\Sigma}_0(\mathbf{x},t) + \boldsymbol{\tau}(\mathbf{x},t) + \mathbf{I}^0(\mathbf{x},t);\end{aligned}$$

→ **this decomposition "does not feel" the state of the comparison solid**

3) Ensemble average (up to two-point statistics) polarizations over all possible realizations at a given point and take the resulting components $\hat{\tau}_r(\mathbf{x},\omega)$ of such average as unknown fields;

4) Average out the strain field solving the balance of linear momentum for each realization at a given point and plug it in the the stationary points of a Hashin & Shtrickman variational principle, which may be written for the $\hat{\tau}_r(\mathbf{x},\omega)$.

⇨ Integral equations for $\hat{\tau}_r(\mathbf{x},\omega)$ ⇨ $< \hat{\tau}(\mathbf{x},\omega) >= \sum_{r=1}^{n} c_r \hat{\tau}_r(\mathbf{x},\omega)$

⇨ Nonlocal stress-strain/state response

⇩

Second-gradient approximation

Ensemble average (up to two-point statistics) - Hashin & Shtrickman v.p.

Integral equation for the **averaged actual polarization-**

Nonlocal stress-strain/state response -Second-gradient approximation

$$<\hat{\boldsymbol{\sigma}}>=<\hat{\boldsymbol{\sigma}}>+<\hat{\boldsymbol{\sigma}}>$$

$$<\hat{\boldsymbol{\sigma}}>(\mathbf{x},\omega)=\left[\hat{\mathbb{C}}_2(\omega)+c_1\left(c_2\tilde{\boldsymbol{\Upsilon}}(\mathbf{0},\omega)+\hat{\mathbb{C}}_1^{2^{-1}}(\omega)\right)^{-1}\right]<\hat{e}>(\mathbf{x},\omega)$$
$$-\frac{c_1c_2}{2}\left(c_2\tilde{\boldsymbol{\Upsilon}}(\mathbf{0},\omega)+\hat{\mathbb{C}}_1^{2^{-1}}(\omega)\right)^{-1}\tilde{\boldsymbol{\Upsilon}}_{,mn}(\mathbf{0},\omega)\left(c_2\tilde{\boldsymbol{\Upsilon}}(\mathbf{0},\omega)+\hat{\mathbb{C}}_1^{2^{-1}}(\omega)\right)^{-1}[\hat{e}_{,nm}(\mathbf{x},\omega)],$$

$$<\hat{\boldsymbol{\sigma}}>(\mathbf{x},\omega)=\left[\mathbb{I}+c_1c_2\hat{\mathbb{C}}_1^2(\omega)\left(\tilde{\boldsymbol{\Upsilon}}^{-1}(\mathbf{0},\omega)+c_2\hat{\mathbb{C}}_1^2(\omega)\right)^{-1}\right]\hat{\mathbf{I}}_2^0(\mathbf{x},\omega)$$
$$+\frac{c_1c_2}{2}\hat{\mathbb{C}}_1^2(\omega)\left(\tilde{\boldsymbol{\Upsilon}}^{-1}(\mathbf{0},\omega)+c_2\hat{\mathbb{C}}_1^2(\omega)\right)^{-1}\tilde{\boldsymbol{\Upsilon}}^{-1}(\mathbf{0},\omega)\tilde{\boldsymbol{\Upsilon}}_{,mn}(\mathbf{0},\omega)\tilde{\boldsymbol{\Upsilon}}^{-1}(\mathbf{0},\omega)\left(\tilde{\boldsymbol{\Upsilon}}^{-1}(\mathbf{0},\omega)+c_2\hat{\mathbb{C}}_1^2(\omega)\right)^{-1}[\hat{\mathbf{I}}^0_{2,nm}(\mathbf{x},\omega)]$$

where

$$\frac{\tilde{\mathbf{\Upsilon}}_{ijkl_{,mn}}(\mathbf{0},\omega)}{H} := \hat{A}_1(\omega)\delta_{ij}\delta_{kl}\delta_{mn} + \hat{A}_2(\omega)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\delta_{mn}$$
$$+ \hat{A}_3(\omega)(\delta_{ij}(\delta_{km}\delta_{ln} + \delta_{kn}\delta_{lm}) + (\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm})\delta_{kl})$$
$$+ \hat{A}_4(\omega)(\delta_{ik}(\delta_{jm}\delta_{ln} + \delta_{jn}\delta_{lm}) + \delta_{il}(\delta_{jm}\delta_{kn} + \delta_{jn}\delta_{km})$$
$$+ \delta_{im}(\delta_{jk}\delta_{ln} + \delta_{jl}\delta_{kn}) + \delta_{in}(\delta_{jk}\delta_{lm} + \delta_{jl}\delta_{km})), \quad (60$$

$H := \int_0^{+\infty} h(r) r dr.$

Isotropic statistics and isotropic phases.
Choose the matrix as a comparison solid

$$\mathbb{G}_0 = \mathbb{G}_2$$

then

$$\hat{A}_1(\omega) = \frac{4}{105}\frac{3\hat{\kappa}(\omega) + \hat{\mu}(\omega)}{\hat{\mu}(\omega)(3\hat{\kappa}(\omega) + 4\hat{\mu}(\omega))}, \quad \hat{A}_2(\omega) = -\frac{1}{35}\frac{3\hat{\kappa}(\omega) + 8\hat{\mu}(\omega)}{\hat{\mu}(\omega)(3\hat{\kappa}(\omega) + 4\hat{\mu}(\omega))}$$
$$\hat{A}_3(\omega) = -\frac{1}{35}\frac{3\hat{\kappa}(\omega) + \hat{\mu}(\omega)}{\hat{\mu}(\omega)(3\hat{\kappa}(\omega) + 4\hat{\mu}(\omega))}, \quad \hat{A}_4(\omega) = \frac{3}{140}\frac{3\hat{\kappa}(\omega) + 8\hat{\mu}(\omega)}{\hat{\mu}(\omega)(3\hat{\kappa}(\omega) + 4\hat{\mu}(\omega))} \quad (61$$

where $\hat{\kappa}(\omega)$, $\hat{\mu}(\omega)$, bulk and shear-like complex moduli.

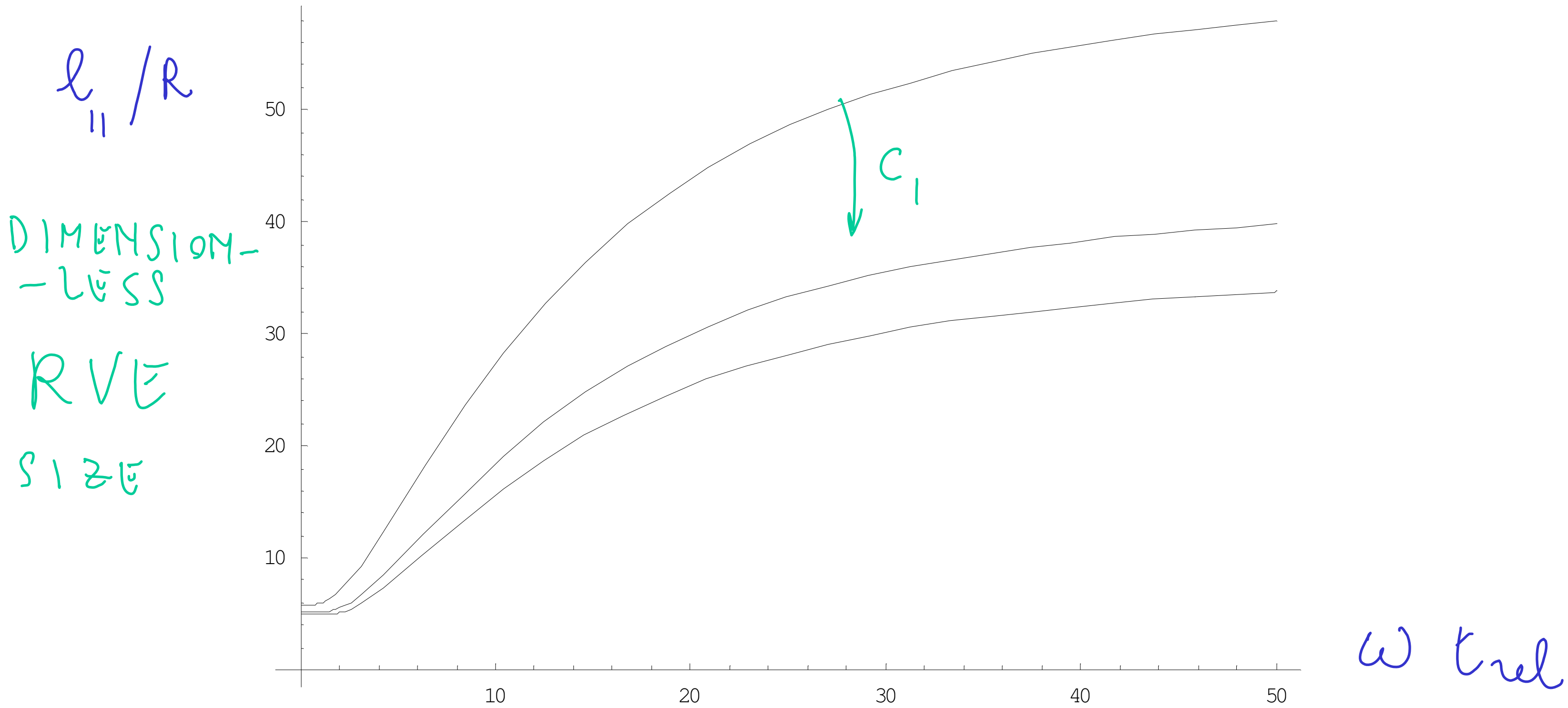
Frequency dependence of the RVE size

Example: Single relaxation type matrix with voids

1) The effect of the residual stress on the RVE

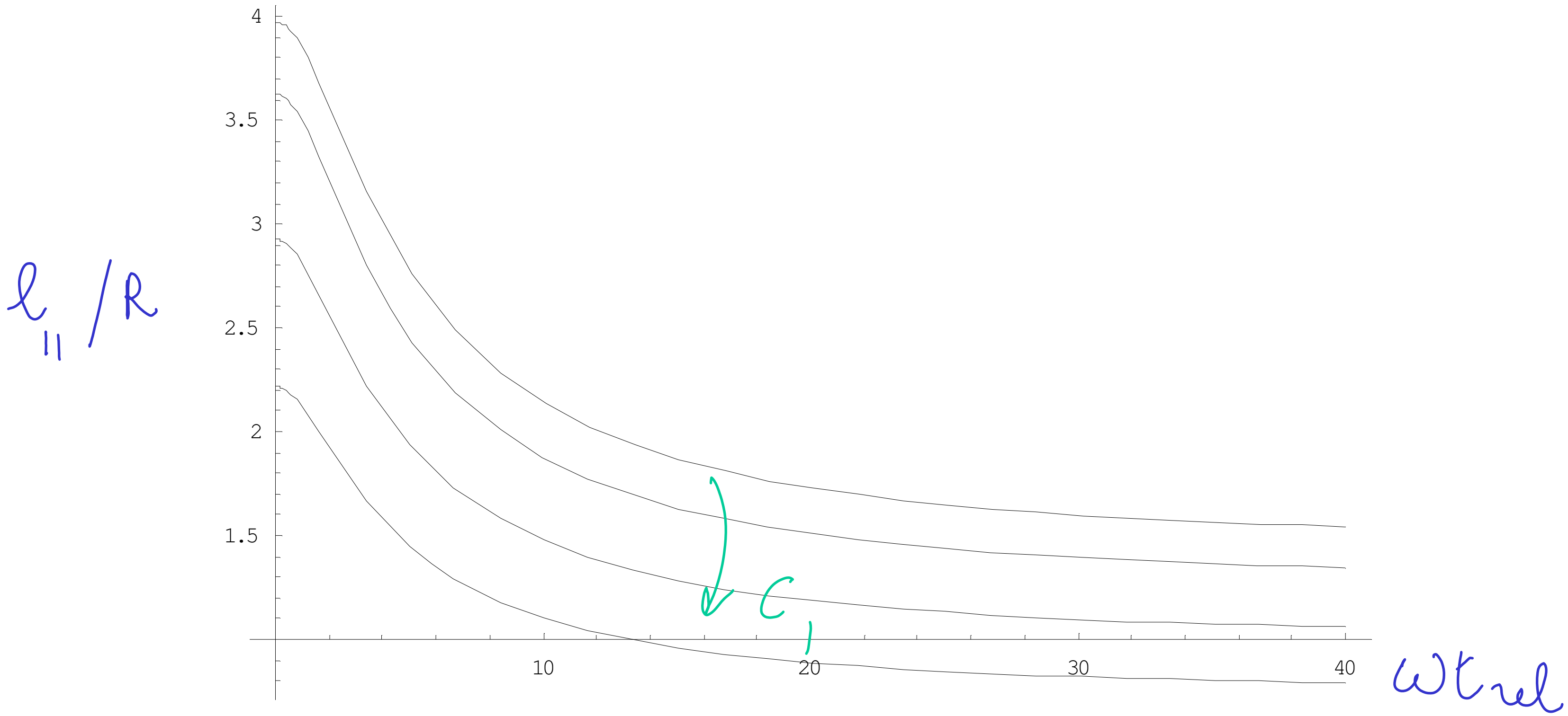
$$\langle \underset{\approx}{\tilde{I}}^{0} \rangle (x_1, \omega) = \hat{i}(\omega) \sin\frac{2\pi x_1}{\ell_{11}} \, \underset{\sim}{e}_1 \otimes \underset{\sim}{e}_1$$

Plot starVOIDS 0.25, 0.2 starVOIDS 0.25, 0.1 starVOIDS 0.25, 0.05 50



1) The effect of the actual strain process on the RVE

In[5]:= Plot ... a 25, 0.2, x ... a 125, 0.2, x ... a 0625, 0.2, x ... a 0313, 0.2, x ... , 40



Out[5]= …Graphics…

For single time relaxation materials there are two competivive effects

What about more realistic materials (Power law relaxation functions)?

Supplementary material

TARGET: $\hat{\boldsymbol{\tau}}(\mathbf{x},\omega)$ **the TFT of the** ***actual polarization***

Balance of l.m.

$$Div(\hat{\mathbb{C}}_0(\omega)\,\hat{e}(\mathbf{x},\omega)) + \hat{f}(\mathbf{x},\omega) + Div(\hat{\boldsymbol{\tau}}(\mathbf{x},\omega) + \hat{\mathbf{I}}^0(\mathbf{x},\omega)) = \mathbf{0}.$$

where $\hat{\Sigma}_0(\mathbf{x},\omega) = \hat{\mathbb{C}}_0(\omega)\,\hat{e}(\mathbf{x},\omega)$ **is the TFT of (12).**

…..we need the one of the random medium though..…

An equation for $\hat{\boldsymbol{\tau}}(\mathbf{x},\omega)$ **comes from balance of l.m.:**

$$\hat{\mathbb{C}}^0(\mathbf{x},\omega)^{-1}\hat{\boldsymbol{\tau}}(\mathbf{x},\omega) + \int_{\mathbb{R}^3} \hat{\mathbf{\Gamma}}_0(\mathbf{x}-\mathbf{x}',\omega)\hat{\boldsymbol{\tau}}(\mathbf{x}',\omega)d\mathbf{x}' = \hat{\boldsymbol{e}}_0(\mathbf{x},\omega) + \hat{\boldsymbol{\epsilon}}(\mathbf{x},\omega)$$

where

$$\hat{\boldsymbol{\epsilon}}(\mathbf{x},\omega) := -\int_{\mathbb{R}^3} \hat{\mathbf{\Gamma}}_0(\mathbf{x}-\mathbf{x}',\omega)\hat{\mathbf{I}}^0(\mathbf{x}',\omega)d\mathbf{x}'$$

$$\frac{\partial^2(\hat{\mathcal{G}}_0(\mathbf{x},\omega))_{jm}}{\partial x_i \partial x_l}\,\hat{\mathbb{C}}_{0_{ijkl}}(\omega) + \delta_{km}\delta(\mathbf{x}) = 0$$

and

$$\hat{\mathbf{\Gamma}}_0(\mathbf{x},\omega) := -\frac{1}{4}\sum_{ijhk}\frac{\partial^2(\hat{\mathcal{G}}_0)_{jh}(\mathbf{x},\omega)}{\partial x_i \partial x_k}(\boldsymbol{e}_i\otimes\boldsymbol{e}_j+\boldsymbol{e}_j\otimes\boldsymbol{e}_i)\otimes(\boldsymbol{e}_h\otimes\boldsymbol{e}_k+\boldsymbol{e}_k\otimes\boldsymbol{e}_h),$$

which is meant to hold in the sense of distributions (see e.g. [9]).

Probabilistic framework

Substituting the trial field for $\hat{\boldsymbol{\tau}}$ and (40) into (28) yield $\hat{e}_0(\mathbf{x},\omega)$:

$$\hat{e}_0(\mathbf{x},\omega) = < \boldsymbol{e} > (\mathbf{x},\omega) - < \hat{\epsilon} > (\mathbf{x},\omega) + \sum_{s=1}^{n} c_s \int_{\mathbb{R}^3} \hat{\boldsymbol{\Gamma}}_0(\mathbf{x}-\mathbf{x}',\omega)\hat{\boldsymbol{\tau}}_s(\mathbf{x}',\omega)d\mathbf{x}' \tag{41}$$

$$< \hat{\epsilon} > (\mathbf{x},\omega) := \sum_{s=1}^{n} c_s \,\hat{\epsilon}_s(\mathbf{x},\omega)$$

$$\hat{\epsilon}_s(\mathbf{x},\omega) := -\int_{\mathbb{R}^3} \hat{\boldsymbol{\Gamma}}_0(\mathbf{x}-\mathbf{x}',\omega)\hat{\mathbf{I}}_s^0(\mathbf{x}',\omega)d\mathbf{x}'. \tag{42}$$

Following [4], the functional (34) in $\hat{\epsilon}_s(\circ,\omega)$ may now be written making use of (38), (40) and $(37)_1$ into (34) and then taking the ensemble average to get

$$\begin{aligned}\mathcal{H}(\{\hat{\boldsymbol{\tau}}_r(\circ,\omega)\}_{r=1,2,\dots n}) \;:=\; & \sum_{r=1}^{n} c_r \int_{\mathbb{R}^3} \{\overline{\hat{\boldsymbol{\tau}}}_r(\mathbf{x},\omega)\cdot(\hat{\mathbb{C}}_r^0)^{-1}(\omega)\hat{\boldsymbol{\tau}}_r(\mathbf{x},\omega) \\ - & \; 2\left[\hat{e}_0(\mathbf{x},\omega) - \sum_{s=1}^{n}\int_{\mathbb{R}^3} \hat{\boldsymbol{\Gamma}}_0(\mathbf{x}-\mathbf{x}',\omega)\hat{\mathbf{I}}_s^0(\mathbf{x}',\omega)\mathcal{P}_{rs}(\mathbf{x}-\mathbf{x}')d\mathbf{x}'\right]\} \\ + & \sum_{r,\,s=1}^{n}\int_{\mathbb{R}^3} \overline{\hat{\boldsymbol{\tau}}}_r(\mathbf{x},\omega)\cdot\left(\int_{\mathbb{R}^3} \hat{\boldsymbol{\Gamma}}_0(\mathbf{x}-\mathbf{x}',\omega)\hat{\boldsymbol{\tau}}_s(\mathbf{x},\omega)\mathcal{P}_{rs}(\mathbf{x}-\mathbf{x}')d\mathbf{x}'\right) d\mathbf{x}\end{aligned}$$

(43

after setting

$$\hat{\mathbb{C}}_r^0(\omega) := \hat{\mathbb{C}}_r(\omega) - \hat{\mathbb{C}}_0(\omega). \tag{44}$$

A BRIEF REVIEW OF RANDOM COMPOSITES[1]

If, at the given ω, the r^{th} phase is homogeneous with complex moduli $\hat{\mathbb{C}}_r(\omega)$, $r = 1, 2, \ldots n$,

$\Rightarrow \hat{\mathbb{C}}(\mathbf{x}, \omega; \alpha)$ the moduli at $\mathbf{x}$ in the α^{th} sample, and its ensemble average, are

$$\hat{\mathbb{C}}(\mathbf{x}, \omega; \alpha) = \sum_{r=1}^{n} \hat{\mathbb{C}}_r(\omega)\chi_r(\mathbf{x}; \alpha) \ \Rightarrow \ <\hat{\mathbb{C}}(\mathbf{x}, \omega)> = \sum_{r=1}^{n} \hat{\mathbb{C}}_r(\omega)\mathcal{P}_r(\mathbf{x}). \tag{37}$$

Ansatz :

$$\hat{\boldsymbol{\tau}}(\mathbf{x}, \omega; \alpha) := \sum_{r=1}^{n} \hat{\boldsymbol{\tau}}_r(\mathbf{x}, \omega)\chi_r(\mathbf{x}; \alpha), \tag{38}$$

$\hat{\boldsymbol{\tau}}_r(\mathbf{x}, \omega)$ is the TFT of $\boldsymbol{\tau}$ when the r^{th} phase is found at $\mathbf{x}$ at the ω.

Take

$$\hat{\mathbf{I}}^0(\mathbf{x}, \omega; \alpha) := \sum_{r=1}^{n} \hat{\mathbf{I}}_r^0(\mathbf{x}, \omega)\chi_r(\mathbf{x}; \alpha); \tag{39}$$

$\Rightarrow$

$$\hat{\epsilon}(\mathbf{x}, \omega; \alpha) := -\sum_{r=1}^{n} \int_{\mathbb{R}^3} \hat{\boldsymbol{\Gamma}}_0(\mathbf{x} - \mathbf{x}', \omega)\hat{\mathbf{I}}_r^0(\mathbf{x}', \omega)\chi_r(\mathbf{x}'; \alpha)d\mathbf{x}' \tag{40}$$

For this case we may follow the reasoning of [10] (see also [4], p.503 relation (22)) where it is shown that

$$\mathcal{P}_{rs}(\mathbf{x}-\mathbf{x}') - c_r c_s = c_r(\delta_{rs} - c_s) h(\mathbf{x}-\mathbf{x}') \tag{50}$$

h is the two-point correlation function.

$\Rightarrow$ (47) becomes:

$$\begin{aligned} & c_r \, (\hat{\mathbb{C}}^0_r)^{-1}(\omega)\hat{\boldsymbol{\tau}}_r(\mathbf{x},\omega) + \sum_{s=1}^{2} c_r(\delta_{rs}-c_s)\int_{\mathbb{R}^3} \hat{\boldsymbol{\Upsilon}}(\mathbf{x}-\mathbf{x}',\omega)\hat{\boldsymbol{\tau}}_s(\mathbf{x}',\omega) h d\mathbf{x}' \\ & = c_r < \tilde{\boldsymbol{e}} > (\xi,\omega) - \sum_{s=1}^{2} c_r(\delta_{rs}-c_s)\int_{\mathbb{R}^3} \hat{\boldsymbol{\Upsilon}}(\mathbf{x}-\mathbf{x}',\omega)\hat{\mathbf{I}}^0_s(\mathbf{x}',\omega) h d\mathbf{x}' \\ & \qquad r = 1,\, 2,\, \ldots n, \end{aligned} \tag{51}$$

where

$$\hat{\boldsymbol{\Upsilon}}(\mathbf{x}-\mathbf{x}',\omega) := \hat{\boldsymbol{\Gamma}}_0(\mathbf{x}-\mathbf{x}',\omega) h(\mathbf{x}-\mathbf{x}').$$

After SFT we get:

$$\tilde{\boldsymbol{\tau}}_r(\xi,\omega) = \sum_{s=1}^{2} c_s \tilde{\mathbb{T}}_{rs}(\xi,\omega)\tilde{\boldsymbol{\Upsilon}}(\xi,\omega)\left(c_r < \tilde{\boldsymbol{e}} > (\xi,\omega) - \sum_{\ell=1}^{2}(\delta_{s\ell} - c_\ell)\tilde{\mathbf{I}}^0_\ell(\xi,\omega)\right),\ r = 1,\, 2, \tag{52}$$

In the $\mathbf{x} - \omega$ domain this is fully nonlocal and has two (potentially competitive) contributions

Frequency dependence of the RVE size

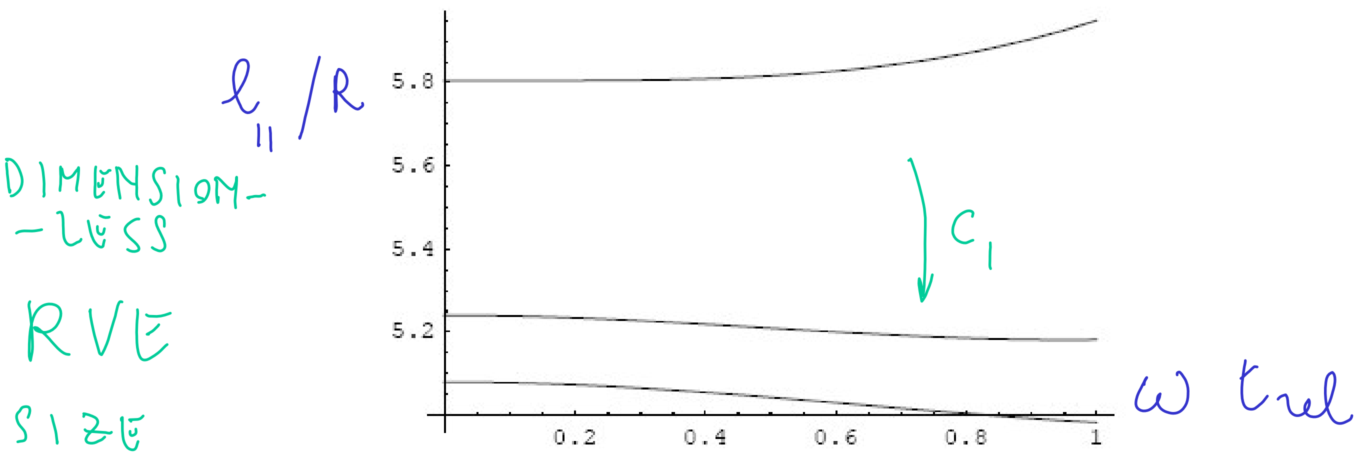
Example: Single relaxation type matrix with voids

1) The effect of the residual stress on the RVE

$$\langle \underline{\underline{\tilde{I}}}^{0} \rangle (x_1, \omega) = \hat{i}(\omega) \sin \frac{2\pi x_1}{\ell_{11}} \; \underset{\sim}{e}_1 \otimes \underset{\sim}{e}_1$$

CHANGING CONCENTRATION OF VOIDS - LOWERING TOWARDS THE "DILUTE CASE"

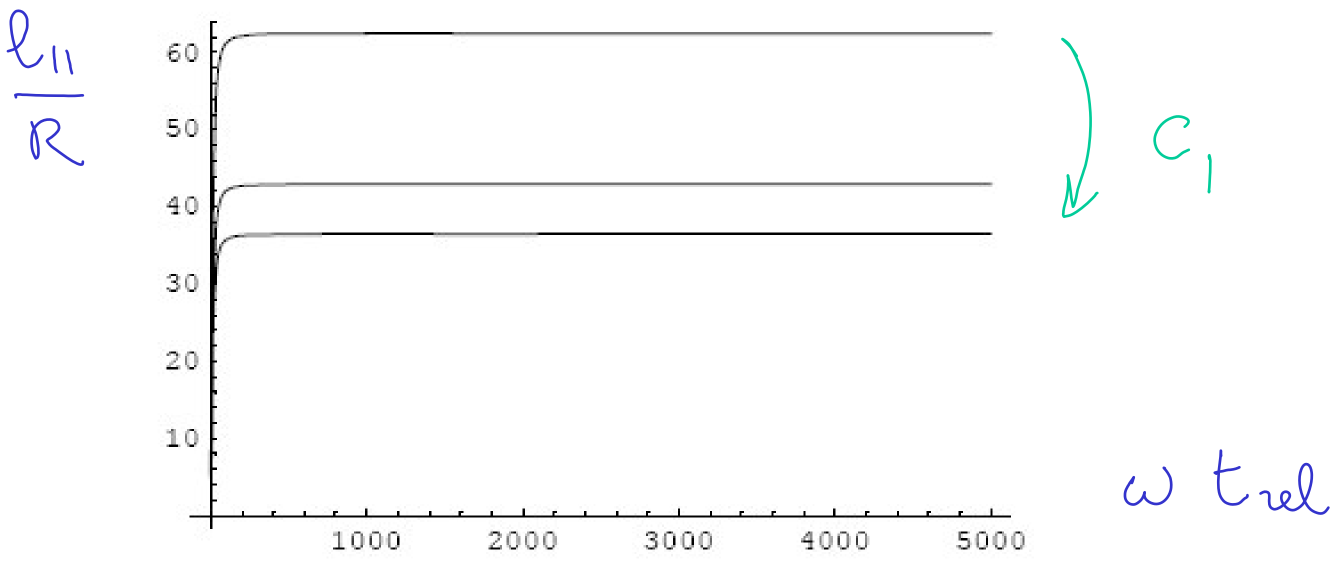
```
Plot[{LL11starVOIDS[x, 0.25, 0.2], LL11starVOIDS[x, 0.25, 0.1],
  LL11starVOIDS[x, 0.25, 0.05]} , {x, 0, 1}]
```



- Graphics -

at lower frequencies (i.e. at equilibrium - i.e. for "slow residual stresses")
the effect of lowering the concentration of voids is to lower the
importance of the nonlocal effect, i.e. to lower the RVE size.

```
Plot[{LL11starVOIDS[x, 0.25, 0.2], LL11starVOIDS[x, 0.25, 0.1],
  LL11starVOIDS[x, 0.25, 0.05]} , {x, 0, 5000}]
```



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Here is the situation at high frequencies : in this case the RVE size decreases as the concentration of voids does.